MATHEMATICS

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Reduced Jordan matrix algebras over complete local rings

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INTRODUCTION

In this paper, we continue our study of Jordan matrix algebras initiated in [12]. The problem we address ourselves to consists in giving necessary and sufficient conditions for two such algebras to be isomorphic, and we solve this problem here for reduced Jordan matrix algebras over complete local rings (see Theorem 4 below). Our approach relies heavily on conjugacy theorems for local idempotents in Jordan pairs which generalize similar results due to Albert and Jacobson ([1] Theorem 9, see also McCrimmon ([6] Theorem 3) and Racine ([13] Lemma 3)) and are established, at least to a considerable extent, in an intrinsic fashion. We also extend Springer's Theorem [14] on isomorphisms between reduced exceptional simple Jordan algebras to this general setting. A peculiar feature of our results is that we do not require any regularity conditions on the coefficient algebras involved. The author is indebted to H.-J. Nastold and W. Scharlau for useful conversations on the subject.

0. NOTATIONS

Let R be a unital commutative associative ring of scalars, remaining fixed throughout this paper. All (linear or quadratic) algebras C over R are supposed to contain an identity, which we shall write as 1, and all homomorphisms of rings and algebras are supposed to take 1 into 1. It will always be assumed that the canonical mapping $R \to C$, $\alpha \mapsto \alpha 1$, is injective. Whenever it is convenient, we shall identify R with $R1$. If invertibility makes sense in C, the set of invertible elements will be denoted by $C^\times$. The (Jacobson) radical of C will be written as rad C. Given an involution J of C, we put $\bar{x}=J(x)$ for $x \in C$ as long as no confusion can arise. J is said to be *central* provided $x\bar{x} \in R1$ for all $x \in C$. In this case, J induces a quadratic form n_C and a linear form t_C on C defined respectively by $x\bar{x} = n_C(x)1$, $x + \bar{x} = t_C(x)1$. If R happens to be a local ring, we write $\mathfrak{m}$ for its maximal ideal and k for its residue class field. The canonical epimorphism from R to k, and, more generally, the reduction modulo $\mathfrak{m}$ of any sensible object over R, will be indicated by the symbol "^". We denote by $\mathbf{N}$ the set of non-negative integers.

1. REDUCED JORDAN MATRIX ALGEBRAS

Let (C, J) be an alternative algebra with central involution over R, $m \geqslant 3$ an integer and $g = \mathrm{diag}(g_1, \ldots, g_m) \in \mathrm{GL}_m(R)$. Then $A = \mathrm{H}_m(C, J, g)$ stands for the R-module of all m-by-m matrices X over C satisfying $X = g{}^t\overline{X}g^{-1}$, where ${}^t\overline{X}$ is the conjugate transpose of X, and having diagonal entries in R. *Throughout this paper it is to be understood that we consider the R-module A only when $m = 3$ or C is associative.* As is well known (see [2], [5], for example), A then carries canonically the structure of a unital quadratic Jordan algebra and is thus called a *reduced Jordan matrix algebra.*

We wish to attach invariants to A and first recall from McCrimmon [8] that, given invertible elements $u, v \in C$, the *u, v-isotope*, $C^{(u,v)}$, of C is the R-module C together with a new product defined by $x \cdot_{u,v} y = (xu)(vy)$ for $x, y \in C$. The algebra $C^{(u,v)}$ is again alternative, with identity element $1^{(u,v)} = (uv)^{-1}$. Also, following [12], the map $J^{(u,v)}\colon C^{(u,v)} \to C^{(u,v)}$ defined by

$$J^{(u,v)}(x) = \bar{x}^{(u,v)} = \overline{x(uv)}(uv)^{-1}$$

for $x \in C$ is a central involution on $C^{(u,v)}$. Another alternative algebra, (D, K), with central involution over R is said to be *isotopic* with (C, J) in case (D, K) becomes isomorphic with $(C^{(u,v)}, J^{(u,v)})$, for some $u, v \in C^{\times}$. This defines an equivalence relation on the class of alternative algebras with central involution ([12] Proposition 2). We shall have occasion to apply the Isotopy Theorem in [12] to reduced Jordan matrix algebras over local rings. In doing so, one arrives at a result which will still be referred to as the

ISOTOPY THEOREM. *Let (C, J), (D, K) be alternative algebras with central involutions over a local ring R and $g, h \in \mathrm{GL}_m(R)$ diagonal matrices of rank $m \geqslant 3$. Then $\mathrm{H}_m(C, J, g)$ and $\mathrm{H}_m(D, K, h)$ are isotopic if and only if (C, J) and (D, K) are isotopic.*

For $m = 3$ it is sometimes important to realize the Jordan algebra structure of $A = \mathrm{H}_3(C, J, g)$ as a special case of a general construction due to McCrimmon [7]. Accordingly, there are a cubic form $N = N_A$ and a quadratic mapping $x \mapsto x^{\#}$ on A such that

(1) $\quad x^3 - T(x)x^2 + Q(x)x - N(x)1 = 0$

in all scalar extensions of A, where $T = T_A$ is the ordinary trace of matrices and $Q = Q_A$ stands for the quadratic form $x \mapsto T(x^\#)$. In the sequel, we refer to Q as the *quadratic trace* of A. To describe this form explicitely, we adopt the customary matrix notation by setting

$$(2) \qquad \alpha[ii] = \alpha e_{ii}, \quad a[ij]_g = g_i a e_{ij} + g_j \bar{a} e_{ji}$$

for $1 \leqslant i, j \leqslant 3$, $i \neq j$, $\alpha \in R$, $a \in C$, where the e_{ii}, e_{ij} are the ordinary matrix units. The elements of A may then be written as

$$(3) \qquad x = \sum_{i=1}^{3} \xi_i[ii] + \sum_{i=1}^{3} x_i[jl]_g$$

with $\xi_i \in R$, $x_i \in C$ $(1 \leqslant i \leqslant 3)$, the second sum being taken over all cyclic permutations (ijl) of (123). Following [7], p. 502, we now obtain

$$Q(x) = \sum_{i=1}^{3} (\xi_j \xi_l - g_j g_l n_C(x_i)).$$

We wish to show that the isometry class of Q is an invariant of A. This poses no problem when R is a field since, in that case, (1) may be characterized intrinsically as being the generic minimum equation for A. Over arbitrary rings, however, the above assertion doesn't seem to be quite obvious.

PROPOSITION 1. *Let (C, J), (D, K) be alternative algebras with central involutions over R, $g, h \in \mathrm{GL}_3(R)$ be diagonal matrices, and put*

$$A = \mathrm{H}_3(C, J, g), \quad B = \mathrm{H}_3(D, K, h).$$

Suppose $\varphi\colon A \to B$ is an isomorphism. Then $N_A = N_B \circ \varphi$, $T_A = T_B \circ \varphi$, $Q_A = Q_B \circ \varphi$.

PROOF. (1) immediately implies

$$(4) \qquad \lambda(x)x^2 - \mu(x)x + \nu(x)1 = 0$$

for every x in every scalar extension of A, where $\lambda(x) = T_A(x) - T_B(\varphi(x))$, $\mu(x) = Q_A(x) - Q_B(\varphi(x))$, $\nu(x) = N_A(x) - N_B(\varphi(x))$. We must show that λ, μ, and ν all vanish identically. Choose first independent indeterminates τ_1, τ_2, τ_3 over R and put $x = \sum \tau_i[ii]$. Expanding (4) yields a homogeneous system of three linear equations in the unknowns $\lambda(x)$, $\mu(x)$, $\nu(x)$ whose (Vandermonde) determinant is $-(\tau_1 - \tau_2)(\tau_1 - \tau_3)(\tau_2 - \tau_3)$. As this obviously is not a zero divisor in $R[\tau_1, \tau_2, \tau_3]$, we may conclude $\lambda(x) = \mu(x) = \nu(x) = 0$. Now let $1 \leqslant i, j \leqslant 3$, $i \neq j$, choose $a \in C$ and put $x = a[ij]_g$. (2) implies $x^2 = g_i g_j n_C(a)(1[ii] + 1[jj])$, and hence, by first looking at the e_{ll}-component $(l \neq i, j)$ and then at the e_{ii}-component of (4), yields $\lambda(x) n_C(a) = 0$, which shows

$$(5) \qquad \lambda(x) = 0$$

at least when a is invertible in C. In the general case, we pass to the ring $R[\eta]$ of dual numbers over R and replace x by $x'=1[ij]_g+\eta x=$ $=(1+\eta a)[ij]_g$ in (5). Comparing coefficients of η yields $\lambda(x)=0$, so that λ vanishes on $C[ij]_g$ and is therefore identically zero. Now (4) collapses to $\mu(x)x=\nu(x)1$, and, writing x as in (3), this amounts to

$$(6)\qquad \mu(x)x_i=0,\ \nu(x)=\xi_i\mu(x)\ \ (1\leqslant i\leqslant 3).$$

If, for some $i=1, 2, 3$, x_i is invertible in C, we obtain $\mu(x)=0$. Otherwise, we pass to the (commutative associative) ring extension $R[\zeta]$ generated by an element ζ subject to the relation $\zeta^3=0$, replace x by $x'=1[jl]_g+\zeta x$ in (6) and arrive at the same conclusion by comparing coefficients of ζ^2. Hence $\mu=0$, and now (6) implies $\nu=0$. This completes the proof.

COROLLARY. *Notations being as in Proposition* 1, *suppose A and B are isomorphic. Then (C, J) and (D, K) are isotopic, and Q_A and Q_B are isometric.*

PROOF. The first assertion follows from the Isotopy Theorem, the second from Proposition 1.

2. JORDAN PAIRS

The appropriate framework for some of our subsequent results is Loos's theory of Jordan pairs [4]. Recall that a *Jordan pair* over R is a pair $V=(V^+, V^-)$ of R-modules together with a pair (Q^+, Q^-) of quadratic mappings $Q^\varepsilon\colon V^\varepsilon \to \mathrm{Hom}_k(V^{-\varepsilon}, V^\varepsilon)$ $(\varepsilon=\pm)$ having the following property. Setting

$$\{xyz\}=L^\varepsilon(x, y)z=Q^\varepsilon(x, z)y=(Q^\varepsilon(x+z)-Q^\varepsilon(x)-Q^\varepsilon(z))y$$

for $x, z\in V^\varepsilon$, $y\in V^{-\varepsilon}$, the identities

$$L^\varepsilon(x, y)Q^\varepsilon(x)=Q^\varepsilon(x)L^{-\varepsilon}(y, x),$$

$$L^\varepsilon(Q^\varepsilon(x)y, y)=L^\varepsilon(x, Q^{-\varepsilon}(y)x),$$

$$Q^\varepsilon(Q^\varepsilon(x)y)=Q^\varepsilon(x)Q^{-\varepsilon}(y)Q^\varepsilon(x)$$

hold in all scalar extensions. Writing Q, L instead of Q^ε, L^ε, for simplicity, V also satisfies (cf. [4] JP20, p. 19)

$$(7)\qquad \left\{\begin{array}{l} Q(\{xyz\})=Q(x)Q(y)Q(z)+Q(z)Q(y)Q(x)+\\ Q(x, z)Q(y)Q(x, z)-Q(Q(x)y, Q(z)y).\end{array}\right.$$

In the sequel, we assume familiarity with the terminology of [4] and, in particular, shall make extensive use of the Peirce decomposition relative to an orthogonal system of idempotents ([4] § 5). If D is such a system, the Peirce (i, j)-subpair relative to D will be written as $V_{ij}(D)$. We also put

$$B(x, y)=id_{V^\varepsilon}-L(x, y)+Q(x)Q(y)$$

for $x \in V^\varepsilon$, $y \in V^{-\varepsilon}$ and recall (cf. [4] 2.12)

$$(8) \qquad B(x, y) = Q(x)Q(x^{-1} - y) = Q(x - y^{-1})Q(y),$$

provided x is invertible in V. The (Jacobson) radical of V will be denoted by rad V. If A is a quadratic Jordan algebra, we write (A, A) for the corresponding Jordan pair. Every orthogonal system X of idempotents in A induces an orthogonal system of idempotents in (A, A), which we shall write as (X, X).

Following [11], a Jordan pair V over R is said to be *connected* in case any two orthogonal local idempotents in $V/\text{rad } V$ are connected in the usual sense. We recall from [11] the following basic result.

CONJUGACY THEOREM. *Let V be a connected Jordan pair over R and $(c_1, \ldots, c_r)$, $(e_1, \ldots, e_r)$ be orthogonal systems of local idempotents in V. Then there exists an inner automorphism h of V such that $hV_2(e_i) = V_2(c_i)$ for $1 \leqslant i \leqslant r$.*

In particular, if V is connected, any two frames in V have the same (finite or infinite) length, called the *capacity* of V. As in [12], V is said to be *complete* if it is connected of finite capacity and

$$V = V_2\Big(\sum_{i=1}^{m} c_i\Big)$$

for every frame $(c_1, \ldots, c_m)$ in V. In any Jordan pair V, we write $I_r(V)$, where $r \geqslant 1$ is an integer, for the collection of orthogonal systems in V consisting of r local idempotents. We finally observe that $R^\times$, the group of units in R, acts canonically on Aut (V), the automorphism group of V, via $(\alpha, h) \mapsto \alpha . h = (\alpha h^+, \alpha^{-1} h^-)$ for $\alpha \in R^\times$ and $h = (h^+, h^-) \in \text{Aut}(V)$.

3. REDUCED CONNECTED JORDAN PAIRS

Let V be a Jordan pair over R. An idempotent $c = (c^+, c^-)$ in V is said to be *reduced* if the canonical maps $R \to Rc^\varepsilon$ determine an isomorphism from (R, R) onto $V_2(c)$, where R is to be viewed as a quadratic Jordan algebra in the natural way. Now suppose $D = (d_1, d_2)$ is an orthogonal system of two reduced idempotents in V. Then we obtain quadratic forms $n_D^\varepsilon \colon V_{12}(D)^\varepsilon \to R$ defined by the relations

$$(9) \qquad Q(x)d_2^{-\varepsilon} = n_D^\varepsilon(x)d_1^\varepsilon$$

for all $x \in V_{12}(D)^\varepsilon$. If $x \in V_{12}(D)^\varepsilon$ happens to be invertible in $V_2(d_1 + d_2)$, the corresponding inverse, which belongs to $V_{12}(D)^{-\varepsilon}$, will be written as x^{-1}. We now collect a few properties of the quadratic forms n_D^ε as follows.

LEMMA 1. *Let V, D be as above and suppose that d_1, d_2 are connected. Then, for $x \in V_{12}(D)^\varepsilon$, $y \in V_{12}(D)^{-\varepsilon}$,*

(i) $Q(x)d_1^{-\varepsilon} = n_D^\varepsilon(x)d_2^\varepsilon$.

(ii) $n_D^{-\varepsilon}(y) = n_D^\varepsilon(Q(d_1^\varepsilon + d_2^\varepsilon)y)$.

(iii) $n_D^\varepsilon(Q(x)y) = n_D^\varepsilon(x)^2 n_D^{-\varepsilon}(y)$.

(iv) *x is invertible in $V_2(d_1+d_2)$ if and only if $n_D^\varepsilon(x)$ is invertible in R, in which case $n_D^\varepsilon(x)^{-1} = n_D^{-\varepsilon}(x^{-1})$.*

PROOF. (i) First suppose x is invertible in $V_2(d_1+d_2)$. Then, by (9), $n_D^\varepsilon(x)$ cannot be a zero divisor in R, and $Q(x)d_1^{-\varepsilon} = \alpha d_2^\varepsilon$ for some $\alpha \in R$. Hence

$$\alpha n_D^\varepsilon(x)d_1^\varepsilon = Q(x)Q(d_2^{-\varepsilon})Q(x)d_1^{-\varepsilon} = Q(Q(x)d_2^{-\varepsilon})d_1^{-\varepsilon} = n_D^\varepsilon(x)^2 d_1^\varepsilon,$$

and (i) follows. If x is arbitrary, choose $u \in V_{12}(D)^\varepsilon$ invertible in $V_2(d_1+d_2)$ and consider the base ring extension $W = R[\zeta] \otimes_R V$, where $R[\zeta]$ is the unital commutative associative R-algebra generated by an element ζ subject to the relation $\zeta^3 = 0$. V may canonically be regarded as an R-subpair of W, and it is easily seen that the orthogonal connected idempotents d_1, d_2 stay reduced in W; also, the corresponding quadratic form on $W_{12}(D)^\varepsilon$ is just the scalar extension of n_D^ε, and $u+\zeta x$ is invertible in $W_2(d_1+d_2)$. Expanding the equation $Q(u+\zeta x)d_1^{-\varepsilon} = n_D^\varepsilon(u+\zeta x)d_2^\varepsilon$ and compairing coefficients of ζ^2, now yields (i) also in the general case. (ii) is obvious, and (iii), (iv) are easy consequences of (i).

In the sequel, it will be important to know how the quadratic form n_D^ε behaves under homomorphisms. The following result, whose trivial proof we shall suppress, describes this behaviour.

LEMMA 2. *Let $\varrho: R \to S$ be a homomorphism of commutative rings. Suppose V, W are Jordan pairs over R, S, respectively, and $h: V \to W$ is a ϱ-semilinear homomorphism. Let $D = (d_1, d_2)$, $D' = (d_1', d_2')$ be orthogonal systems of reduced connected idempotents in V, W, respectively, and $\delta_i^\varepsilon \in S^\times$, $i = 1, 2$, such that $h^\varepsilon d_i^\varepsilon = \delta_i^\varepsilon d_i'^\varepsilon$. Then*

$$n_{D'}^\varepsilon \circ h^\varepsilon|_{V_{12}(D)^\varepsilon} = \delta_1^\varepsilon \delta_2^\varepsilon(\varrho \circ n_D^\varepsilon).$$

For the rest of this section we shall assume that R is a local ring. A Jordan pair over R is said to be *reduced* in case all its local idempotents are reduced. Consider now a reduced connected Jordan pair V over R. For $D = (d_1, d_2)$, $D' = (d_1', d_2') \in I_2(V)$, the Conjugacy Theorem yields an inner automorphism h of V and $\delta_i^\varepsilon \in R^\times$, $i = 1, 2$, such that $h^\varepsilon d_i^\varepsilon = \delta_i^\varepsilon d_i'^\varepsilon$. Using Lemma 2 we obtain $n_{D'}^\varepsilon(h^\varepsilon x) = \delta_1^\varepsilon \delta_2^\varepsilon n_D^\varepsilon(x)$ for all $x \in V_{12}(D)^\varepsilon$, hence

$$n_{D'}^+(h^+x)n_{D'}^-(h^-y) = n_D^+(x)n_D^-(y)$$

for all $x \in V_{12}(D)^+$, $y \in V_{12}(D)^-$. This and Lemma 1 (iv) show that the scalars $n_D^+(x)n_D^-(y)$, with x, y varying over the elements of $V_{12}(D)^+$, $V_{12}(D)^-$, respectively, which are invertible in $V_2(d_1+d_2)$, generate a subgroup of $R^\times$ which is independent of the choice of D. This subgroup is called the *norm group* of V and is denoted by N, or N_V to indicate dependence on V. The quotient $R^\times/N$ is written as Γ or Γ_V.

PROPOSITION 2. *Let V be a reduced connected Jordan pair over R, $m \geqslant 2$ an integer and $D=(d_1, \ldots, d_m) \in I_m(V)$. For $1 \leqslant i, j \leqslant m$, $i \neq j$, put $D(i,j)= =(d_i, d_j) \in I_2(V)$ and $n_{ij}^\varepsilon = n_{D(i,j)}^\varepsilon$. Then, given $x \in V_{ij}(D)^+$ invertible in $V_2(d_i+d_j)$, the image $\omega_{ij}(D)$ of $n_{ij}^+(x)$ in Γ is independent of x, and we have the relations*

$$\omega_{ij}(D) = \omega_{ji}(D),$$

$$\omega_{il}(D) = \omega_{ij}(D)\omega_{jl}(D),$$

the latter for $i, j, l \in \{1, \ldots, m\}$ mutually distinct.

PROOF. If $x, x' \in V_{ij}(D)^+$ are both invertible in $V_2(d_i+d_j)$, we apply Lemma 1 (iv) and obtain $n_{ij}^+(x') = n_{ij}^+(x')n_{ij}^-(x^{-1})n_{ij}^+(x) \equiv n_{ij}^+(x) \bmod N$, which shows that, indeed, $\omega_{ij}(D)$ does not depend on x. The first of the subsequent relations being obvious, we turn to the second and choose $x \in V_{ij}(D)^+$, $y \in V_{jl}(D)^+$ invertible in $V_2(d_i+d_j)$, $V_2(d_j+d_l)$, respectively. Then $z = \{x d_j^- y\} \in V_{il}(D)^+$, and (7) in conjunction with standard Peirce multiplication rules yields $Q(z)d_l^- = Q(x)Q(d_j^-)Q(y)d_l^-$, hence

$$n_{il}^+(z) = n_{ij}^+(x) n_{il}^+(y).$$

This completes the proof.

According to Proposition 2, the elements $\omega_{ij}(D)$ $(1 \leqslant i, j \leqslant m,\ i \neq j)$ are completely determined by the elements $\omega_{1i}(D)$ $(2 \leqslant i \leqslant m)$. We therefore put

$$\omega(D) = (\omega_{1i}(D))_{2 \leqslant i \leqslant m} \in \Gamma^{m-1}$$

and call this the *norm class* of D (in Γ). As before, we shall write ω_V instead of ω to indicate dependence on V. We wish to show that, under suitable restrictions, the norm class of D determines its orbit under the automorphism group. However, to illuminate the concepts introduced above, we first turn to Jordan pairs associated with Jordan matrix algebras.

PROPOSITION 3. *Let (C, J) be an alternative algebra with central involution over R, $m \geqslant 3$ an integer, $g = \operatorname{diag}(g_1, \ldots, g_m) \in \mathrm{GL}_m(R)$, and put $A = \mathrm{H}_m(C, J, g)$. Then the Jordan pair $V=(A, A)$ is reduced as well as complete, and $N_V = n_C(C^\times)$. Setting $d_i = (1[ii], 1[ii])$ for $1 \leqslant i \leqslant m$, $D= =(d_1, \ldots, d_m)$ is a frame in V such that*

$$\omega_{ij}(D) = g_i g_j \bmod n_C(C^\times)$$

for $1 \leqslant i, j \leqslant m$, $i \neq j$.

PROOF. The completeness of V follows from [12] Proposition 4 and implies that V is reduced. Clearly, D is a frame in V; also, for i, j as indicated and $a \in C$, we have $a[ij]_g^2 = g_i g_j n_C(a)(d_i^\varepsilon + d_j^\varepsilon)$, which implies $n_{ij}^\varepsilon(a[ij]_g) = g_i g_j n_C(a)$. The rest of the proof is now obvious.

Proposition 3 shows that our approach is an intrinsic generalization of the one adopted for the same purpose by Albert and Jacobson [1]. It also follows either from Proposition 2 or from the proof of Proposition 3 that, for a reduced connected Jordan pair V of capacity $\geqslant 3$ over R and $D=(d_1, d_2) \in I_2(V)$, the elements of N_V have the form $n_D^+(x)n_D^-(y)$ with x, y in $V_{12}(D)^+$, $V_{12}(D)^-$, respectively, invertible in $V_2(d_1+d_2)$. This fact, however, will not be used in the sequel. Returning now to the abstract situation, we first require the following elementary observation.

LEMMA 3. *Let V be a reduced connected Jordan pair over R and $D=(d_1, d_2) \in I_2(V)$. Then every $\varrho \in N$ admits an inner automorphism h of $V_2(d_1+d_2)$ satisfying $hV_2(d_i)=V_2(d_i)$ and $h^+d_i^+=\varrho d_i^+$ for $i=1, 2$.*

REMARK. Note that every such h extends trivially to an inner automorphism of V which is the identity on $V_0(d_1+d_2)$.

PROOF. We may of course assume $V=V_2(d_1+d_2)$. Given $x \in V_{12}(D)^+$, $y \in V_{12}(D)^-$ both invertible in V, (8) yields $B(x, x^{-1}-y)=Q(x)Q(y)$, $B(x^{-1}-y, x)=Q(y)Q(x)$, and these maps are bijective. Hence $(x, x^{-1}-y)$ is quasi-invertible and $B(x, x^{-1}-y)d_i^+=n_D^+(x)n_D^-(y)d_i^+$ for $i=1, 2$. Lemma 3 follows.

We are now ready to prove

THEOREM 1. *Let V be a reduced connected Jordan pair over R, $m \geqslant 3$ an odd integer and $D, E \in I_m(V)$. Then the following statements are equivalent.*

(i) *D and E are conjugate under* Aut (V).

(ii) $\omega(D)=\omega(E)$.

(iii) *D and E are conjugate under* $R^\times$. Inn (V).

PROOF. "(iii) $\Rightarrow$ (i)" is obvious, and "(i) $\Rightarrow$ (ii)" follows from Lemma 2. It therefore remains to show that (ii) implies (iii). Proposition 2 gives $\omega_{ij}(D)=\omega_{ij}(E)$, $1 \leqslant i, j \leqslant m$, $i \neq j$, and if we set $D=(d_1, \ldots, d_m)$, $E=(e_1, \ldots, e_m)$, the Conjugacy Theorem yields an inner automorphism h of V as well as $\delta_l^\varepsilon \in R^\times$, $1 \leqslant l \leqslant m$, such that $h^\varepsilon e_l^\varepsilon=\delta_l^\varepsilon d_l^\varepsilon$. Then, for any such h, we obtain, by choosing $x \in V_{ij}(E)^+$ invertible in $V_2(e_i+e_j)$ and by applying Lemma 2, $n_{D(i,j)}^+(h^+x)=\delta_i^+\delta_j^+n_{E(i,j)}^+(x)$. Since $\omega_{ij}(E)=\omega_{ij}(D)$, and as N contains $(R^\times)^2$, this means

$$(10) \qquad \delta_i^+(\delta_j^+)^{-1} \in N \quad (1 \leqslant i,\ j \leqslant m,\ i \neq j).$$

We now claim: *For $s=1, \ldots, m-1$, there exists $h_s \in$ Inn (V) satisfying $h_sV_2(e_i)=V_2(d_i)$ $(1 \leqslant i \leqslant m)$ and $h_s^+e_i^+=\delta_1^+d^+$ $(1 \leqslant i \leqslant s)$.* This is shown by induction on s. For $s=1$, $h_s=h$ does the job. Suppose $s<m-1$ and h_s has the desired properties. Then, for some $\gamma \in R^\times$, $h_s^+e_{s+1}^+=\gamma d_{s+1}^+$ and, by (10), $\varrho=\delta_1^+\gamma^{-1}$ belongs to N. Hence Lemma 3 and its accompanying remark yield an inner automorphism g of V satisfying $gV_2(d_i)=V_2(d_i)$ $(1 \leqslant i \leqslant m)$, $g^+d_{s+1}^+=\varrho d_{s+1}^+$, $g^+d_{s+2}^+=\varrho d_{s+2}^+$, and $g^+d_i^+=d_i^+$ $(1 \leqslant i \leqslant m$, $i \neq s+1$, $s+2)$. It follows that $h_{s+1}=gh_s$ satisfies our requirements for $s+1$ in place

of s, which completes the induction. Rephrasing our claim for $s=m-1$, we see that there are $h'\in \operatorname{Inn}(V)$, $\alpha, \beta\in R^{\times}$ such that $h'V_2(e_i)=V_2(d_i)$ $(1\leqslant i\leqslant m)$, $h'^{+}e_i^{+}=\alpha d_i^{+}$ $(1\leqslant i\leqslant m-1)$, $h'^{+}e_m^{+}=\beta d_m^{+}$, and, again by (10), we have $\beta\alpha^{-1}\in N$. Now observe that $t=\frac{1}{2}(m-1)\geqslant 1$ is an integer. Given $j=1, \ldots, t$, Lemma 3 leads to some $f_j\in \operatorname{Inn}(V)$ satisfying $f_jV_2(d_i)=V_2(d_i)$ $(1\leqslant i\leqslant m)$, $f_jd_i=d_i$ $(1\leqslant i\leqslant m,\ i\neq j,\ j+t)$, $f_j^{+}d_j^{+}=\beta\alpha^{-1}d_j^{+}$, $f_j^{+}d_{j+t}^{+}=\beta\alpha^{-1}d_{j+t}^{+}$. Hence $\beta^{-1}\cdot f_1\ldots f_th'\in R^{\times}$. $\operatorname{Inn}(V)$ sends E onto D, which completes the proof of the theorem.

It would of course be desirable to establish Theorem 1 also in the case when m is even. This we have been unable to do. Instead we settle for the following alternative.

THEOREM 2. *Let V be a complete reduced Jordan pair without extreme radical over R and of capacity $\geqslant 3$. Then, for frames D, E in V, the following statements are equivalent.*

(i) *D and E are conjugate under* $\operatorname{Aut}(V)$.

(ii) $\omega(D)=\omega(E)$.

PROOF. Write m for the capacity of V. We must prove that (ii) implies (i). By Theorem 1, we may assume that m is even, so, in particular, $m\geqslant 4$. Hence the Coordinatization Theorem leads to associative algebras (C, J), (C', J') with central involutions over R and to matrices $g=\operatorname{diag}(g_1, \ldots, g_m)$, $g'=\operatorname{diag}(g_1', \ldots, g_m')\in \mathrm{GL}_m(R)$ such that $g_1=g_1'=1$ and the following holds: Setting $A=\mathrm{H}_m(C, J, g)$, $A'=\mathrm{H}_m(C', J', g')$, there are isomorphisms $V\to(A, A)$, $V\to(A', A')$ sending D, E onto the standard diagonal frames X, X' of (A, A), (A', A'), respectively. Now put $h=\operatorname{diag}(g_1, \ldots, g_m, 1)$, $h'=\operatorname{diag}(g_1', \ldots, g_m', 1)$, $B=\mathrm{H}_{m+1}(C, J, h)$, $B'=\mathrm{H}_{m+1}(C', J', h')$ and denote by Y, Y' the standard diagonal frames in (B, B), (B', B'), respectively. By the Isotopy Theorem, the Jordan pairs (B, B), (B', B') are isomorphic and so have the same norm group. Furthermore, by Proposition 3, $\omega(Y)=\omega(Y')$, and hence, as $m+1$ is odd, Theorem 1 produces an isomorphism from (B', B') onto (B, B) sending Y' to Y. Restricting this map to (A', A') yields an automorphism of V which sends E to D.

4. COMPLETENESS

Let A be a Jordan algebra over R (not assumed to be local) and $\mathfrak{C}$ a descending chain of ideals in A. There evidently is a unique way of giving A the structure of a topological Jordan algebra over R, $R=R1$ carrying the induced topology, such that $\mathfrak{C}$ makes up a fundamental system of neighborhoods of 0 in A. The corresponding topology will be called the *$\mathfrak{C}$-adic topology*. We are particularly interested in the following special case of this general construction. Suppose $\mathfrak{S}$ is an ideal in A. Following McCrimmon [9], we define the *Penico series* of *$\mathfrak{S}$ in A* inductively by

$$\mathfrak{S}_A^{(0)}=\mathfrak{S},\ \mathfrak{S}_A^{(n+1)}=P(\mathfrak{S}_A^{(n)})A\quad (n\in \mathbf{N}),$$

where P stands for the quadratic representation of A. This is a descending chain $\mathfrak{C}$ of ideals, and we shall speak of the $\mathfrak{s}$-*adic* rather than the $\mathfrak{C}$-adic topology in this context. If $\mathfrak{m}$ is an ideal in R containing 2, it may happen that the $\mathfrak{m}A$-adic topology defined above is strictly finer than the usual $\mathfrak{m}$-adic topology on A defined by the descending chain $(\mathfrak{m}^n A)_{n\in\mathbf{N}}$. For this reason, descending chains of ideals more general than Penico series will be discussed in the sequel. If $\mathfrak{s}$ is an ideal, a descending chain $\mathfrak{C}=(\mathfrak{r}_n)_{n\in\mathbf{N}}$ of ideals is said to *dominate* $\mathfrak{s}$ provided $\mathfrak{r}_0=\mathfrak{s}$ and $\mathfrak{r}_n \supset \mathfrak{s}_A^{(n)}$ for all $n \in \mathbf{N}$.

PROPOSITION 4. *Let $\varrho: R \to S$ be a homomorphism of commutative rings and $h: A \to B$ a ϱ-semilinear epimorphism of Jordan algebras A, B over R, S, respectively. Suppose we are given a descending chain $\mathfrak{C}$ of ideals in A dominating the kernel of h such that A is Hausdorff and complete relative to the $\mathfrak{C}$-adic topology. Let $(c_1', \ldots, c_m')$ be an orthogonal system of idempotents in B. Then there exists an orthogonal system $(c_1, \ldots, c_m)$ of idempotents in A such that $hc_i=c_i'$ for $1\leqslant i\leqslant m$. Also, for every such system, h induces canonically a ϱ-semilinear isomorphism from*

$$A_2(c_i)/\mathrm{rad}\, A_2(c_i) \text{ onto } B_2(c_i')/\mathrm{rad}\, B_2(c_i').$$

In particular, c_i is local if and only if c_i' is local.

The problem of lifting the idempotents c_i', $1\leqslant i\leqslant m$, as indicated easily reduces to the case $m=1$ and may then be solved along the lines of the proof of Proposition 5 in [10]. The remainder of Proposition 4 is a straight forward consequence of the fact that the radical of A contains $\mathfrak{s}=\ker h$. To see this, let $x\in\mathfrak{s}$. For integers $n\geqslant 1$, $l\geqslant 2^n$ we have

$$x^l=P(x^{2^{n-1}})x^{l-2^n},$$

which shows $x^l\in\mathfrak{s}_A^{(n)}$ for all $n\geqslant 0$, $l\geqslant 2^n$ by induction on n. Hence x is topologically nilpotent in the $\mathfrak{C}$-adic topology, and so $\mathfrak{s}$ turns out to be a quasi-invertible ideal, as desired.

For the rest of this section, R is assumed to be a *complete* local ring. We recall from [15] VIII, proof of Theorem 5, that, if E is a finitely generated R-module, E is complete relative to the $\mathfrak{m}$-adic topology. We now require two more auxiliary results, the first one of these being concerned with quadratic forms. Such a form is said to be *non-defective* if its radical agrees with the radical of the induced bilinear form (which happens automatically in case 2 is not a zero divisor). The following Hensel Lemma is a trivial modification of the "Bilinear Lemma" in [15] VIII § 7, p. 278.

LEMMA 4. *Let q be a quadratic form over R defined on a finitely generated R-module E. Suppose $a\in E$, $\alpha\in R$ satisfy $q(a)\equiv\alpha \bmod \mathfrak{m}$. Then, if $-1\in\hat{q}(\hat{a},\hat{E})$, there exists some $a_1\in E$ such that $a\equiv a_1 \bmod \mathfrak{m}E$ and $q(a_1)=\alpha$. In particular, this holds true when $\hat{q}$ is non-defective.*

Next we turn to the behaviour of Jordan matrix algebras under reduction mod $\mathfrak{m}$.

PROPOSITION 5. *Let (C, J) be an alternative algebra with central involution over R and suppose C is finitely spanned as an R-module. Let $g \in \mathrm{GL}_m(R)$ $(m \geqslant 3)$ be a diagonal matrix and put $A = \mathrm{H}_m(C, J, g)$. Then*

$$\mathfrak{m}C \subset \mathrm{rad}\, C, \ \mathrm{rad}\, \hat{C} = (\mathrm{rad}\, C)/\mathfrak{m}C,$$

$$\mathfrak{m}A \subset \mathrm{rad}\, A = \mathrm{H}_m(\mathrm{rad}\, C, J, g), \ \mathrm{rad}\, \hat{A} = (\mathrm{rad}\, A)/\mathfrak{m}A.$$

Finally, the central involution induced by J on $C/\mathrm{rad}\, C$ being denoted by J', the Jordan algebras $A/\mathrm{rad}\, A$ and $\mathrm{H}_m(C/\mathrm{rad}\, C, J', \hat{g})$ over k are canonically isomorphic.

Here $\mathfrak{s} = \mathrm{H}_m(\mathrm{rad}\, C, J, g)$ is supposed to consist of all matrices in A whose entries belong to $\mathrm{rad}\, C$.

PROOF. As C is complete relative to the $\mathfrak{m}$-adic topology, $\mathfrak{m}C$ is contained in $\mathrm{rad}\, C$. This implies $\mathrm{rad}\, \hat{C} = (\mathrm{rad}\, C)/\mathfrak{m}C$, and, by the same token, we have $\mathfrak{m}A \subset \mathrm{rad}\, A$, $\mathrm{rad}\, \hat{A} = (\mathrm{rad}\, A)/\mathfrak{m}A$. Clearly $\mathfrak{m}A \subset \mathfrak{s}$, and $A/\mathfrak{s}$ becomes canonically isomorphic with $\mathrm{H}_m(C/\mathrm{rad}\, C, J', \hat{g})$. Hence, as $C/\mathrm{rad}\, C$ has finite dimension over k, $A/\mathfrak{s}$ is a simple algebra, which gives $\mathrm{rad}\, A \subset \mathfrak{s}$. Conversely, let $x \in \mathfrak{s}$. Since $(\mathrm{rad}\, C)/\mathfrak{m}C = \mathrm{rad}\, \hat{C}$ is a nilpotent ideal, we have $(\mathrm{rad}\, C)^n \subset \mathfrak{m}C$ for some integer $n \geqslant 1$. Therefore x^n has entries in $\mathfrak{m}C$ and so belongs to $\mathfrak{m}A$. This shows that $\mathfrak{s}$ defines a nil ideal in $\hat{A}$ and implies $\mathfrak{s} \subset \mathrm{rad}\, A$. The proof of Proposition 5 is now complete.

Note that, (C, J) being as above, $C/\mathrm{rad}\, C$ is either a composition algebra or, for char $k = 2$, a purely inseparable exponent one extension field of k (cf. Kaplansky [3]). With this in mind, we can now establish the following reduction theorem.

THEOREM 3. *Let R be a complete local ring and (C, J), (D, K) alternative algebras with central involutions over R such that D is finitely spanned as an R-module and $D/\mathrm{rad}\, D$ is a composition algebra over k. Suppose $g, h \in \mathrm{GL}_m(R)$ are diagonal matrices of rank $m \geqslant 3$ and the Jordan algebras $A = \mathrm{H}_m(C, J, g)$, $B = \mathrm{H}_m(D, K, h)$ are isotopic. Let $\mathfrak{m}A \subset \mathfrak{s}$, $\mathfrak{m}B \subset \mathfrak{t}$ be ideals in A, B, respectively, and suppose we are given a descending chain $\mathfrak{C}$ of ideals in B dominating $\mathfrak{t}$ such that B is Hausdorff and complete relative to the $\mathfrak{C}$-adic topology. Then, if $A/\mathfrak{s}$ and $B/\mathfrak{t}$ are isomorphic over k, A and B are isomorphic over R.*

PROOF. We put $A' = A/\mathfrak{s}$, $B' = B/\mathfrak{t}$, $V = (A, A)$, $W = (B, B)$, $V' = (A', A')$, $W' = (B', B')$ and employ the terminology of the previous section. As V and W are isomorphic, we have $N_V = N_W$, which we write as N; similarly, $N_{V'} = N_{W'} = N'$. We also put $\Gamma = \Gamma_V = \Gamma_W$, $\Gamma' = \Gamma_{V'} = \Gamma_{W'}$ and, since $\hat{N} = N'$, obtain induced homomorphisms $\hat{}: \Gamma^s \to \Gamma'^s$ for every $s \geqslant 1$. Now choose an isomorphism φ from A' onto B', denote by $X = (d_1, \dots, d_m)$ the standard diagonal frame in A, write X' for its image in A', and put

$Y' = \varphi(X')$. According to Proposition 4, Y' may be lifted to a frame $Y = (e_1, \ldots, e_m)$ in B, and Lemma 2 yields $\omega_V(\mathfrak{X})^{\wedge} = \omega_W(\mathfrak{Y})^{\wedge}$, where $\mathfrak{X} = (X, X)$, $\mathfrak{Y} = (Y, Y)$. Given $i, j = 1, \ldots, m$ distinct and $x \in A_{ij}(X)$, $y \in B_{ij}(Y)$ invertible in $A_2(d_i + d_j)$, $B_2(e_i + e_j)$, respectively, this amounts to the following assertion: There are an integer $n \geqslant 1$ and $u_l, v_l \in A_{ij}(X)$ $(1 \leqslant l \leqslant n)$ all invertible in $A_2(d_i + d_j)$ such that $n^{+}_{\mathfrak{Y}(i,j)}(y) \equiv \alpha \bmod \mathfrak{m}$, where

$$\alpha = n^{+}_{\mathfrak{X}(i,j)}(x) \prod_{l=1}^{n} n^{+}_{\mathfrak{X}(i,j)}(u_l) n^{-}_{\mathfrak{X}(i,j)}(v_l).$$

Since $D/\operatorname{rad} D$ is a composition algebra over k, $n^{+}_{\mathfrak{Y}(i,j)}$ becomes non-defective after reduction modulo $\mathfrak{m}$, and so Lemma 4 applies, which shows $\omega_V(\mathfrak{X}) = \omega_W(\mathfrak{Y})$. Now Theorem 2 produces an isomorphism $\Phi = (\Phi^+, \Phi^-)$: $W \to V$ sending $\mathfrak{Y}$ to $\mathfrak{X}$. Hence $\Phi^+ : B \to A$ is an isotopy which sends Y to X and so, in particular, preserves identities. Thus it must be an isomorphism, and the proof is complete.

As an immediate consequence of Theorem 3 we shall now derive the surprising fact that the classification problem for *arbitrary* reduced Jordan matrix algebras over a complete local ring is equivalent to the classification problem for *simple* reduced Jordan matrix algebras over the residue class field.

THEOREM 4. *Let (C, J), (D, K) be alternative algebras with central involutions over a complete local ring R satisfying $\cap\, \mathfrak{m}^n = 0$ such that D is finitely spanned as an R-module and $D/\operatorname{rad} D$ is a composition algebra over k. Suppose $g, h \in \mathrm{GL}_m(R)$ are diagonal matrices of rank $m \geqslant 3$ and put $A = \mathrm{H}_m(C, J, g)$, $B = \mathrm{H}_m(D, K, h)$. Then, if A and B are isotopic, the following statements are equivalent.*

(i) *A and B are isomorphic (over R).*

(ii) *$A/\mathfrak{m}A$ and $B/\mathfrak{m}B$ are isomorphic (over k).*

(iii) *$A/\operatorname{rad} A$ and $B/\operatorname{rad} B$ are isomorphic (over k).*

PROOF. "(i) $\Rightarrow$ (iii)" is obvious, and, in "(iii) $\Rightarrow$ (ii)", Proposition 5 shows that we may assume $\mathfrak{m} = \{0\}$, i.e., $R = k$. Then we put $\mathfrak{s} = \operatorname{rad} A$, $\mathfrak{t} = \operatorname{rad} B$ in Theorem 3 and observe that, as $\operatorname{rad} B$ is Penico solvable (see Theorems 1, 2 and Penico's Theorem in McCrimmon [9]), B is complete relative to its $(\operatorname{rad} B)$-adic topology. Finally, "(ii) $\Rightarrow$ (i)" follows by setting $\mathfrak{s} = \mathfrak{m}A$, $\mathfrak{t} = \mathfrak{m}B$, $\mathfrak{C} = (\mathfrak{m}^n B)_{n \in \mathbf{N}}$ in Theorem 3.

5. SPRINGER'S THEOREM

The most effective result on the isomorphism problem for reduced exceptional simple Jordan algebras is due to Springer [14] (see also Racine [13], McCrimmon [6]). It says that two such algebras are isomorphic if and only if they have isomorphic coefficient algebras and isometric quadratic traces. The machinery developed in this paper allows us to establish the following extension of Springer's Theorem.

THEOREM 5. *Notations being as in Theorem 4, let us assume $m=3$. Then $A=\mathrm{H}_3(C, J, g)$ and $B=\mathrm{H}_3(D, K, h)$ are isomorphic if and only if (C, J) and (D, K) are isotopic and Q_A and Q_B are isometric.*

PROOF. These conditions are clearly necessary, by the corollary of Proposition 1. Conversely, assume (C, J), (D, K) are isotopic and Q_A, Q_B are isometric. Then A and B are isotopic, and, by means of Proposition 5, a straight forward computation yields

$$\mathrm{H}_3(\operatorname{rad} C, J, g)=\{x \in A : Q_A(x) \in \mathfrak{m},\ Q_A(x, A) \subset \mathfrak{m}\}$$

as well as an analogous equation for B. Hence it follows, by reducing modulo radicals and consulting Proposition 5 again, that $A/\operatorname{rad} A$ and $B/\operatorname{rad} B$ have isometric quadratic traces. Also, as isotopy between finite dimensional semi-simple alternative algebras with central involutions reduces to isomorphism, they have isomorphic coefficient algebras. Hence, by Springer's Theorem, $A/\operatorname{rad} A$ and $B/\operatorname{rad} B$ are isomorphic. Now Theorem 4 implies that A and B are isomorphic.

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